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9 Technical Report No. 7

6 BOUNDING THEOREMS FOR STRESS INTENSITY FACTORS.

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Bounding theorems for the changes in elastic energy due to a crack in a finite body and crack tip Stress Intensity Factors are established by use of the variational principles of the theory of elasticity. It is shown that the change in elastic energy due to a crack in a finite body is larger than the change in energy due to the same crack in an infinite body with the same boundary conditions when stresses are prescribed on the external boundary, and smaller when displacements are prescribed. The energy changes can be expressed as functions of the crack tip Stress Intensity Factors and for special loadings. (over)		

Abstract, continued

bounds for single mode Stress Intensity Factors are obtained. The obtained inequalities are in agreement with known numerical solutions of finite cracked bodies.

## 1. Introduction

The present work is concerned with new bounding theorems in fracture mechanics. The theorems are established by the use of the variational Principles of the Theory of Elasticity. The theorems show the influence of the finite size of the cracked body on the change in elastic energy due to the crack as a function of the prescribed boundary conditions. For special loadings, when single modes occur, it is shown that the stress intensity factors (abbreviated SIF) of a finite cracked body with prescribed stresses on the external boundary will always be larger than the crack tip SIF of an infinite cracked body with the same boundary conditions and that the SIF of a finite cracked body with prescribed displacements on the external boundary will be always smaller than the SIF of an infinite cracked body with the same boundary conditions.

## 2. The Bounding Procedure

The problem of bounding the change in elastic energy due to a crack in a finite body will here be considered in terms of the classical extremum principles of minimum potential and complementary energies. These principles have been previously used for bounding the effective elastic moduli of heterogeneous materials [1] and recently for bounding effective moduli of cracked material [2].

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Finite Body with Prescribed Stresses.

The change in energy due to a crack in a finite body with prescribed stresses on the external boundary will be bounded by use of the principle of minimum potential energy. Consider the finite body of volume  $V_1$  containing a stress free crack (fig. 1a). Traction boundary conditions are prescribed on the external boundary  $S_1$ .

$$T_i(S_1) = T_i^a \quad (1)$$

The potential energy of the cracked body is

$$U_{P1} = \frac{1}{2} \int_{V_1} \sigma_{ij1} \epsilon_{ij1} dV - \int_{S_1} T_i^a u_{i1} dS \quad (2)$$

where  $\sigma_{ij}$ ,  $\epsilon_{ij}$  and  $u_i$  are the unknown stresses, strains and displacements in the body. The potential energy stored in the cracked body is given by,

$$U_{P1}^o = U_{P1}^o - \delta U_1 \quad (3)$$

where  $U_{P1}^o$  is the potential energy of the body of volume  $V_1$  subject to (1) in the absence of the crack, and is a function of  $\sigma_{ij1}^o$ ,  $\epsilon_{ij1}^o$  the stresses and strains in the uncracked body,

$$U_{P1}^o = - \frac{1}{2} \int_{V_1} \sigma_{ij1}^o \epsilon_{ij1}^o dV \quad (4)$$

and  $\delta U$  is the change in potential energy due to the crack and according to a well known result, Eshelby [3], can be expressed as

$$\delta U = \frac{1}{2} \sigma_{ij}^0 \int_{\text{on the crack}} [u_i] n_j dS \quad (5)$$

where  $[u_i]$  is the displacement jump across the crack faces. From (5) it is seen that only the displacement in the immediate vicinity of the crack has to be known to compute  $\delta U$ .

The potential energy functional is defined by:

$$\bar{U}_{Pl} = \frac{1}{2} \int_{V_1} \tilde{\sigma}_{ij} \tilde{\epsilon}_{ij} dV - \int_{S_1} T_i^a \tilde{u}_i dS \quad (6)$$

where  $u_i$  is an admissible displacement field which must merely be continuous in the region excluding the crack. Since in the present problem boundary tractions are prescribed  $\tilde{u}_i$  is not restricted by boundary conditions. The "stresses" and "strains",  $\tilde{\sigma}_{ij}$  and  $\tilde{\epsilon}_{ij}$  are defined by

$$\tilde{\epsilon}_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (a)$$

(7)

$$\tilde{\sigma}_{ij} = C_{ijkl} \tilde{\epsilon}_{kl} \quad (b)$$

The extremum principle states that

$$\tilde{U}_{p1} \geq U_{p1} \quad (8)$$

Thus the problem of bounding  $\delta U_1$  reduces to that of constructing an appropriate admissible displacement field. To do this consider another larger finite body of volume  $V$  including  $V_1$  and containing the same crack, where

$$V = V_1 + V_2 > V_1 \quad (9)$$

and apply the tractions  $T_1^a$  to the external boundary  $S_2$

$$T_i(S_2) = T_i^a \quad (10)$$

The displacement solution to this problem is  $u_{i2}$ . The part of this displacement field contained in the region  $V_1$  of the larger body is an admissible displacement field for the body of volume  $V_1$  subjected to (1), since it is continuous within the volume  $V_1$ , excluding the crack.

$$\tilde{u}_i = u_{i2}(\text{in } V_1) \quad (11)$$

The displacement field  $u_{i2}$  can be expressed as the sum of the displacement  $u_{i2}^0$  in the body  $V$  without the crack and the perturbation  $u_{i2}^1$  due to the presence of the crack.

$$u_{i2} = u_{i2}^0 + u_{i2}^1 \quad (12)$$



The strains and stresses associated with  $u_{12}$  are

$$\epsilon_{1j2} = \epsilon_{1j2}^o + \epsilon'_{1j2} \quad (a)$$

(13)

$$\sigma_{1j2} = \sigma_{1j2}^o + \sigma'_{1j2} \quad (b)$$

Thus

$$\tilde{\epsilon}_{1j} = \epsilon_{1j2}(\text{in } V_1) \quad (a)$$

(14)

$$\tilde{\sigma}_{1j} = \sigma_{1j2}(\text{in } V_1) \quad (b)$$

The potential energy of the larger body is defined as

$$U_{P2} = \frac{1}{2} \int_{V_1+V_2} \sigma_{1j2} \epsilon_{1j2} dV - \int_{S_2} T_1^a u_{12} dS \quad (15)$$

and can be expressed as

$$U_{P2} = -\frac{1}{2} \int_{V_1+V_2} \sigma_{1j2}^o \epsilon_{1j2}^o dV - \delta U_2 \quad (16)$$

In order to bound the real potential energy of the finite body of volume  $V_1$ , the expression  $\bar{U}_p$  is defined by

$$\bar{U}_p = -\frac{1}{2} \int_{V_1} \sigma_{1j2}^o \epsilon_{1j2}^o dV - \delta U_2 \quad (17)$$

Comparing (16) and (17) it is easy to see that  $\bar{U}_p$  takes the form

$$\bar{U}_p = U_{p2} + \frac{1}{2} \int_{V_2} \sigma_{ij2}^o \epsilon_{ij2}^o dV \quad (18)$$

By substituting (11) and (14) into (6) the potential energy functional for  $V_1$  takes the form:

$$\bar{U}_{p1} = \frac{1}{2} \int_{V_1} \sigma_{ij2} \epsilon_{ij2} dV - \int_{S_1} T_1^a u_{12} dS \quad (19)$$

It will now be shown that

$$\bar{U}_p \geq \bar{U}_{p1} \quad (20)$$

from which it then follows by (8) that

$$\bar{U}_p \geq U_{p1} \quad (21)$$

Assuming (20) to be correct it can be written in the form

$$\begin{aligned} \frac{1}{2} \int_{V_1+V_2} \sigma_{ij2} \epsilon_{ij2} dV - \int_{S_2} T_1^a u_{12} dS + \frac{1}{2} \int_{V_2} \sigma_{ij2}^o \epsilon_{ij2}^o dV &\geq \frac{1}{2} \int_{V_1} \sigma_{ij2} \epsilon_{ij2} dV - \\ &- \int_{S_1} T_1^a u_{12} dS \end{aligned} \quad (22)$$

or after rearranging

$$\frac{1}{2} \int_{V_2} (\sigma_{ij2} \epsilon_{ij2} + \sigma_{ij2}^0 \epsilon_{ij2}^0) dV - \int_{S_2} T_i^a u_{i2} dS + \int_{S_1} T_i^a u_{i2} dS \geq 0 \quad (23)$$

It will now be assumed that  $T_i^a$  has the special form

$$T_i^a = \sigma_{ij}^0 n_j \quad (24)$$

both on  $S_1$  and on  $S_2$ . This is known as homogeneous traction boundary conditions and the elasticity stress solutions in this case are

$$\sigma_{ij1}^0 = \sigma_{ij2}^0 = \sigma_{ij}^0 \quad (25)$$

for homogeneous elastic bodies of arbitrary shape. The associate strain and displacements are

$$\epsilon_{ij1}^0 = \epsilon_{ij2}^0 = \epsilon_{ij}^0 = S_{ijkl} \sigma_{kl}^0 \quad (26)$$

$$u_{i1}(x) = u_{i2}(x) = u_i(x) = \epsilon_{ij}^0 x_j$$

It now follows from virtual work that

$$- \int_{S_2} T_i^a u_{i2} dS + \int_{S_1} T_i^a u_{i2} dS = - \int_{V_2} \sigma_{ij}^0 \epsilon_{ij2} dV \quad (27)$$

Introducing (27) together with (13) and (25) into (22) we obtain

$$\int_{V_2} \left( -\frac{1}{2} \sigma_{ij}^0 \epsilon_{ij2}' + \frac{1}{2} \sigma_{ij2}' \epsilon_{ij}^0 + \frac{1}{2} \sigma_{ij2}' \epsilon_{ij2}' \right) dV \geq 0 \quad (28)$$

By the symmetry of the  $C_{ijkl}$  with respect to  $ij,kl$  interchange

$$\sigma_{ij}^0 \epsilon_{ij2}' = \sigma_{ij2}' \epsilon_{ij}^0 \quad (29)$$

and (28) reduces to

$$\int_{V_2} \frac{1}{2} \sigma_{ij2}' \epsilon_{ij2}' dV \geq 0 \quad (30)$$

Evidently (30) is correct as the integrand is positive definite since it is an elastic strain energy density. This proves the inequality (20), therefore also the inequality (21). Expressing  $\bar{U}_p$  by (17) and  $U_{p1}$  by (3) and (4) and using (25), (21) becomes:

$$-\frac{1}{2} \int \sigma_{ij}^0 \epsilon_{ij}^0 dV - \delta U_2 \geq -\frac{1}{2} \int_{V_1} \sigma_{ij}^0 \epsilon_{ij}^0 dV - \delta U_1 \quad (31)$$

and therefore

$$\delta U_1 \geq \delta U_2 \quad (32)$$

In words: when the same homogeneous stress boundary conditions are prescribed on the external boundary of two finite bodies, the energy change

due to the presence of a crack will be larger in the smaller body.

A limiting case of the problem considered is when the larger body of volume  $V$  becomes infinite. It follows that the change in energy due to a crack in a finite body subjected to stress boundary conditions is always larger than the energy change in an infinite body subjected to the same boundary conditions. Thus Eq. (32) can be extended to read

$$\delta U_1 \geq \delta U_2 \geq \delta U^\infty \quad (33)$$

Finite body with prescribed displacements.

The change in elastic energy due to a crack in a finite body with prescribed displacements on its external boundary will be bounded by use of the principle of minimum complementary energy. The procedure is very similar to that followed above, but instead of an admissible displacement field an admissible stress field has to be constructed.

Let us prescribe homogeneous displacements on the external boundaries  $S_1$  and  $S_2$ , of the two finite cracked bodies of volumes  $V_1$  and  $V=V_1+V_2$  illustrated in fig.1.

$$u_i^a(S_1) = \epsilon_{ij}^0 x_j \quad (a)$$

(34)

$$u_i^a(S_2) = \epsilon_{ij}^0 x_j \quad (b)$$

The stress, strain and displacement fields in the two bodies are  $\sigma_{ij1}$ ,  $\epsilon_{ij1}$ ,  $u_{i1}$  and  $\sigma_{ij2}$ ,  $\epsilon_{ij2}$ ,  $u_{i2}$ , respectively, in the presence of the crack and  $\sigma_{ij1}^0$ ,  $\epsilon_{ij1}^0$ ,  $u_{i1}^0$  and  $\sigma_{ij2}^0$ ,  $\epsilon_{ij2}^0$ ,  $u_{i2}^0$  in the absence of the crack. The complementary energies of the two bodies are

$$U_{C1} = \frac{1}{2} \int_{V_1} \sigma_{ij1} \epsilon_{ij1} dV - \int_S T_{i1} u_i^a dS = - \frac{1}{2} \int_{V_1} \sigma_{ij1}^0 \epsilon_{ij1}^0 dV + \delta U_1 \quad (35)$$

$$U_{C2} = \frac{1}{2} \int_{V_1+V_2} \sigma_{ij2} \epsilon_{ij2} dV - \int_{S_2} T_{i2} u_i^a dS = - \frac{1}{2} \int_{V_1} \sigma_{ij2}^0 \epsilon_{ij2}^0 dV + \delta U_2 \quad (36)$$

where

$$T_i = \sigma_{ij} n_j \quad (37)$$

As in the previous case the energy changes  $\delta U$  are given by (5).

The stress solution to the larger body, of volume  $V$ , over the part  $V_1$  of this volume is an admissible stress field for the problem of the body  $V_1$ , since it satisfies equilibrium everywhere in  $V_1$  and has no displacement boundary conditions to satisfy, thus

$$\tilde{\sigma}_{ij} = \sigma_{ij2} \text{ (in } V_1) \quad (38)$$

The potential energy functional for the body  $V_1$  is defined as

$$\bar{U}_{C1} = \frac{1}{2} \int_{V_1} \tilde{\sigma}_{ij} \tilde{\epsilon}_{ij} dV - \int_{S_1} \tilde{T}_i u_1^a dS \quad (39)$$

where

$$\tilde{T}_i = \tilde{\sigma}_{ij} n_j \quad (40)$$

and  $\tilde{\epsilon}_{ij}$  is related to  $\tilde{\sigma}_{ij}$  by (7b).

The principle of minimum complementary energy states that

$$\bar{U}_{C1} \geq U_{C1} \quad (41)$$

Define  $\bar{U}_C$  as

$$\bar{U}_C = -\frac{1}{2} \int_{V_1} \sigma_{ij2}^o \epsilon_{ij2}^o dV + \delta U_2 \quad (42)$$

From (36) it follows that  $\bar{U}_C$  can be expressed as

$$\begin{aligned} \bar{U}_C = U_{C2} + \frac{1}{2} \int_{V_2} \sigma_{ij2}^o \epsilon_{ij2}^o dV &= \frac{1}{2} \int_{V_1+V_2} \sigma_{ij2} \epsilon_{ij2} dV - \int_{S_2} T_{i2} u_1^a dS + \\ &+ \frac{1}{2} \int_{V_2} \sigma_{ij2}^o \epsilon_{ij2}^o dV \end{aligned} \quad (43)$$

For boundary conditions (34), (25-26) are true for elastic homogeneous bodies of arbitrary shape. Expressing the admissible stresses and the related strains by (13) and using the principle of virtual work for the volume  $V_2$  it can be shown, as in the previous case that

$$\bar{U}_C \geq \tilde{U}_C \quad (44)$$

It then follows that

$$\bar{U}_C \geq U_{C1} \quad (45)$$

From (35) and (43) substituted into (45) we obtain

$$\delta U_2 \geq \delta U_1 \quad (46)$$

In words: When the same homogeneous displacement boundary conditions are prescribed on the external boundaries of two finite bodies, the energy change due to the presence of a crack will be smaller in the smaller body.

Again, a limiting case is obtained when  $V$  becomes infinite: the change in elastic energy due to a crack contained in a finite elastic body with prescribed displacements on its external boundary is always smaller than the change in elastic energy due to the same crack in an infinite body with the same prescribed displacements at infinity. Eq. (46) extends to read:



$$\delta U^\infty \geq \delta U_2 \geq \delta U_1 \quad (47)$$

### 3. Bounding of SIF

In the preceding general bounds on the changes in elastic energies due to the presence of a crack in a finite body were obtained. These bounds can be transformed into bounds on SIF for special loadings.

The change in energy due to a Griffith crack imbedded in an isotropic body as given by (5) can be expressed as a function of the crack tip SIF. For a 3-dimensional body

$$\delta U = \frac{2\pi}{E} \int_0^a [K_I^2(a) + K_{II}^2(a) + (1-\nu)K_{III}^2(a)] da \quad (48)$$

When the cracked body is 2-dimensional, only Modes I and II exist and

$$\delta U = \frac{2\pi}{E} \int_0^a [K_I^2(a) + K_{II}^2(a)] da \quad (49)$$

where  $a$  is half the crack length.

When the cracked body becomes infinite the stress intensity factors are

$$K_I = \sigma_{22}^0 a^{1/2} \quad (a)$$

$$K_{II} = \sigma_{12}^0 a^{1/2} \quad (b) \quad (50)$$

$$K_{III} = \sigma_{23}^0 a^{1/2} \quad (c)$$

where  $\sigma_{22}^0$ ,  $\sigma_{12}^0$ ,  $\sigma_{23}^0$  are the tensile and shear stresses applied at infinity, when stress boundary conditions are prescribed, or

$$K_I = E \epsilon_{22}^0 a^{1/2} \quad (a)$$

$$K_{II} = G \epsilon_{12}^0 a^{1/2} \quad (b) \quad (51)$$

$$K_{III} = G \epsilon_{23}^0 a^{1/2} \quad (c)$$

when displacements of the kind  $u_i^0 = \epsilon_{ij}^0 x_j$  are applied on the external boundary.

For the general case of mixed mode loading, only the combination  $[K_I^2 + K_{II}^2 + (1-\nu)K_{III}^2]$  of the stress intensity factors can be bounded. For special single mode loadings bounds on single SIF can be obtained. A few examples will be considered in the following.

#### Example 1:

##### a) Stress boundary conditions.

Consider a finite plane rectangle of dimensions  $2b$  and  $2c$  containing a crack of length  $2a$ , (fig.2). with prescribed boundary conditions on the external edges of the form:

$$\begin{aligned}
\sigma_{22}(x_2 = \pm c) &= \sigma^0 \\
\sigma_{11}(x_1 = \pm b) &= 0 \\
\sigma_{12}(x_1 = \pm b) &= \sigma_{12}(x_2 = \pm c) = 0
\end{aligned}
\tag{52}$$

Since the loading is symmetric this is a mode I loading, and the only SIF existent is  $K_I$ . The change in elastic energy due to the crack is

$$\delta U = \frac{2\pi}{E} \int_0^a K_I^2(a) da
\tag{53}$$

The change in elastic energy due to a crack in an infinite body with the same boundary conditions is

$$\delta U^\infty = \frac{2\pi}{E} \int_0^a \sigma_0^2 a da = \frac{2\pi}{E} \int_0^a K_I^{\infty 2}(a) da
\tag{54}$$

Substituting (53) and (54) into the inequality (33) we obtain

$$\int_0^a K_I^2(a) da \geq \int_0^a K_I^{\infty 2}(a) da
\tag{55}$$

This inequality will hold for any crack length only when the same inequality holds for the integrands, therefore

$$K_I^2 \geq K_I^{\infty 2}
\tag{56}$$

or

$$K_I \geq \sigma^0 a^{1/2} \quad (57)$$

b. Displacement boundary conditions

Consider again the rectangle of Fig 2. with prescribed displacement boundary conditions on the external edges of the rectangle of the form:

$$\begin{aligned} u_2 (x_2 = \pm c) &= \pm v_0 \\ u_1 (x_2 = \pm c) &= 0 \\ u_2 (x_1 = \pm b) &= u_1 (x_1 = \pm b) = 0 \end{aligned} \quad (58)$$

This deformation is symmetric so only the mode I SIF will enter the energy change expression. We can express the prescribed displacement  $v_0$  as

$$v_0 = \epsilon^0 c$$

The stress intensity factor for an infinite body with prescribed displacements at infinity according to (51a), is

$$K_I^\infty = E \epsilon^0 a^{1/2} \quad (59)$$

When the changes in elastic energy for the finite and infinite rectangle are expressed as a function of the mode I SIF and inequality (47) is used the following bound on  $K_I(a)$  is obtained:

$$K_I(a) \leq K_I^\infty(a) \quad (60)$$

or

$$K_I(a) \leq \frac{E v_0}{c} a^{1/2} \quad (61)$$

Isida [7] calculated, numerically, the mode I SIF for a finite rectangle with various boundary conditions. For boundary conditions (51) his results satisfy (57), i.e. the SIF are always larger than the one for an infinite cracked body.

A second problem considered by Isida was that of the finite rectangle with the following boundary conditions

$$\begin{aligned} u_2(x_2 = \pm c) &= \pm v_0 & (a) \\ u_1(x_2 = \pm c) &= 0 & (b) \\ \sigma_{11}(x_1 = \pm b) &= \sigma_{12}(x_1 = \pm b) = 0 & (c) \end{aligned} \quad (62)$$

It should be pointed out that the results for this case cannot be bounded by the theorems developed in the present work. Since the boundary conditions are mixed [(62c) are traction boundary conditions]. However these results also satisfy (61), i.e.  $K_I$  is always smaller than  $\frac{E\nu_0}{c} a^{1/2}$ .

It should be emphasized that there are no similar inequalities for the SIF of a finite body with mixed boundary conditions. It may be either larger or smaller than the SIF for the corresponding infinite body. Isida [4] also considered the case of the rectangle with the following loading

$$\begin{aligned} u_2(x_2 = \pm c) &= \pm v_0 \\ \sigma_{12}(x_1 = \pm b) &= \sigma_{12}(x_2 = \pm c) = 0 \\ \sigma_{11}(x_1 = \pm b) &= 0 \end{aligned} \quad (63)$$

For this case the results obtained were sometimes larger than  $K_I$  and sometimes smaller, depending on the ratios  $a/b$  and  $c/b$ .

#### Example 2:

A second example considered will be that of a finite rectangle of dimensions  $b$  and  $2h$  containing a crack of length  $a$ , (fig.3).

The corresponding infinite body for this problem will be that of a

half plane containing a crack. The mode I SIF for the infinite body is

$$K_I^\infty = 1.12 \sigma^0 a^{1/2} \quad (64)$$

when a stress  $\sigma^0$  perpendicular to the crack is prescribed at infinity and

$$K_I = 1.12 E \epsilon^0 a^{1/2} \quad (65)$$

when a displacement field

$$u_2^0 = \epsilon^0 x_2 \quad (66)$$

is prescribed at infinity.

Various solutions to the problem of the finite rectangle with boundary conditions

$$\begin{aligned} \sigma_{22} (x_2 = \pm h) &= \sigma^0 \\ \sigma_{12} (x_1 = 0, b) &= \sigma_{12} (x_2 = \pm h) = 0 \\ \sigma_{11} (x_1 = 0, b) &= 0 \end{aligned} \quad (67)$$

have been summarized in [8] and the  $K_I(a)$  obtained are always larger than  $1.12 \sigma^0 a^{1/2}$  as predicted by eq. (56).

Also the mode III loading with prescribed stresses for a finite bar with an edge crack was solved numerically, [5], and the results are also as expected: the obtained  $K_{III}(a)$  are larger than  $K_{III}^{\infty}$ .

#### 4. Conclusions

By using the variational principles of the theory of elasticity it has been shown that the change in energy due to a crack in a finite body increases as the dimensions of the body decrease and is always larger than the change in energy due to a crack in an infinite body, when stress boundary conditions are prescribed. When displacement boundary conditions are prescribed the inverse is true. As the dimensions of the body decrease, the change in elastic energy also decreases and the change in energy due to a crack in an infinite body is always larger than in a finite one.

The changes in elastic energy can be expressed as a function of the crack tip SIF and for special loadings the SIF can be shown to be always larger (for prescribed stresses) or smaller (for prescribed displacements) than the SIF of a crack in an infinite body.

The changes in elastic energy can be related to the crack criticality criterion. The decrease in the size of a finite body will precipitate failure when stresses are applied and postpone it when displacements are applied. When alternating stresses are applied to a finite fatigue specimen



and a crack is growing during the fatigue process, the crack tip SIF increases with the dimensions of the crack and so does the crack criticality.

Existing solutions to crack problems in finite bodies have been shown to be in agreement with the theorems developed in the present work.

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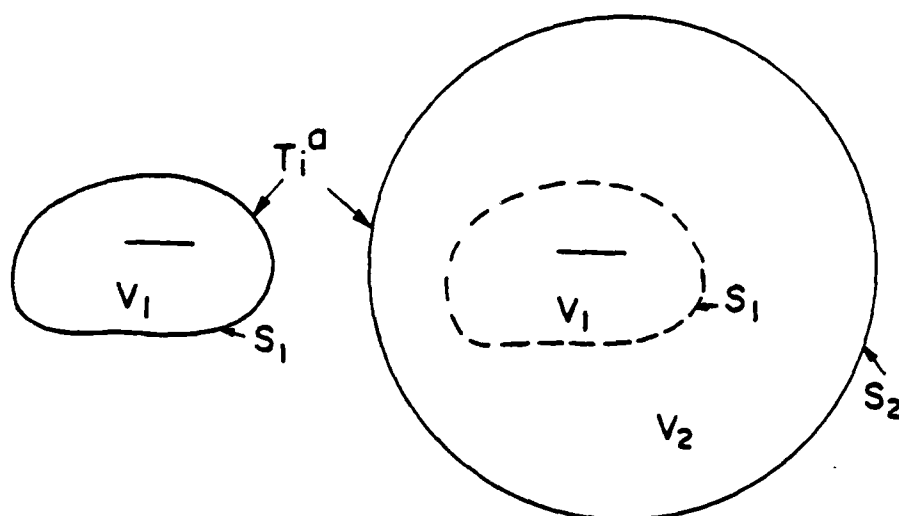


FIG. 1

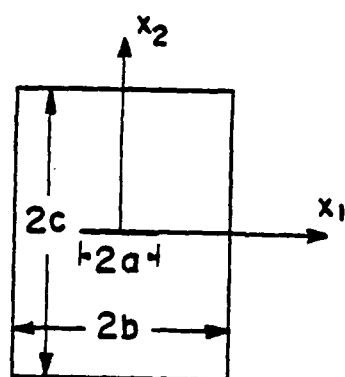


FIG. 2

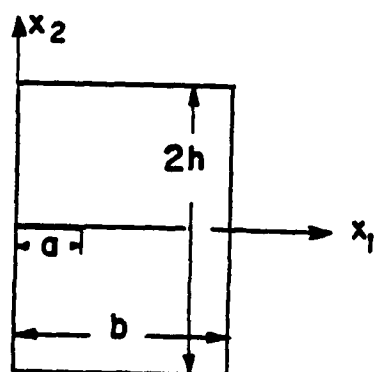


FIG. 3

Figure Captions

Fig. 1: Finite cracked bodies of arbitrary size

Fig. 2: Finite rectangle with central crack

Fig. 3: Finite rectangle with edge crack

